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University Of Diyala – College Of Engineering

Digital Signal Processing

Course Instructor
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Lecture No. 5

Third Class
Department of Computer and Software Engineering

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Lecture Outline

- Continuous Time Fourier Series
- Discrete time Fourier series
- Discrete Fourier Transform (DFT)
- Fast Fourier Transform (FFT)
- Decimation in time Fast Fourier Transform
- Decimation in frequency Fast Fourier Transform

Fourier Series Representation

Fourier Series:

- Fourier series allows any periodic waveform in time to be decomposed into a sum of sine and cosine waveforms. The first requirement in realising the FS is to calculate the fundamental period, T, which is the shortest time over which the signal repeats.
- For a periodic signal with fundamental period T sec, the FS represents this signal as a sum of sine and cosine components that are harmonics of the fundamental frequency $f_0 = 1/T$ Hz.

Fourier Series Representation

- The Fourier series can be written in a number of different ways:

$$x(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right)\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left[A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right]$$

$$= \sum_{n=0}^{\infty}\left[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right] \qquad (1)$$

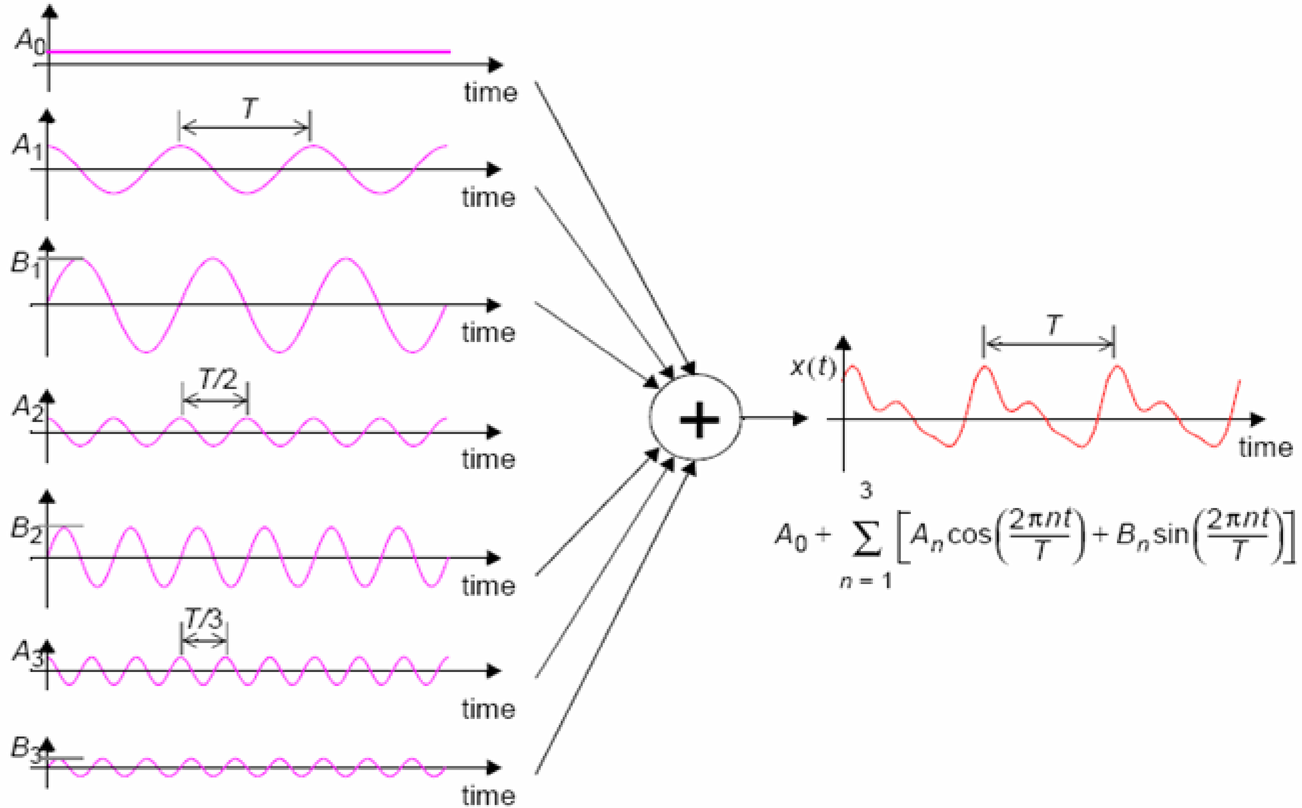
$$= A_0 + A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t) + A_3 \cos(3\omega_0 t) + \ldots$$
$$B_1 \sin(\omega_0 t) + B_2 \sin(2\omega_0 t) + B_3 \sin(3\omega_0 t) + \ldots$$

Where A_n and B_n are the amplitudes of the cos and sin waveforms, $\omega_0 = 2\pi f_0$ rad /sec is angular frequency.

Fourier Series Representation

- In more descriptive language, the above Fourier Series says that any periodic signal can be reproduced by adding a (possibly infinite!) series of harmonically related sinusoidal waveforms of amplitudes A_n or B_n.
- Therefore, *if a periodic signal with a fundamental period of say 0.01 sec is identified, then the Fourier Series will allow this waveform to be represented as a sum of various cosine and sine waves at frequencies of 100 Hz (fundamental frequency), 200 Hz, 300 Hz (Harmonics) and so on. The amplitudes of these components are given by A0, A_1, B_1, A_2, B_2 … and so on.*
- So, how are the values of A_n and B_n calculated??

Fourier Series Representation



Fourier series for a periodic signal $x(t)$. If we analyse a periodic signal and realizethe cosine and sine wave Fourier coefficients of appropriate amplitudes A_n and B_n, then summing these components will lead to exactly the original signal.

Fourier Series Representation

For that, we multiply both sides of (1) by cos ($p\omega_0 t$) where p is any arbitrary positive integer, then we get:

$$\cos(p\omega_0 t)x(t) = \cos(p\omega_0 t)\sum_{n=0}^{\infty}\left[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right] \qquad \textbf{(2)}$$

Integrating Eq: (2) over one period, T, we get:

$$\int_0^T \cos(p\omega_0 t)x(t)dt = \int_0^T \left\{\cos(p\omega_0 t)\sum_{n=0}^{\infty}\left[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right]\right\}dt$$

$$= \boxed{\sum_{n=0}^{\infty}\int_0^T \{A_n \cos(p\omega_0 t)\cos(n\omega_0 t)\}dt} + \boxed{\sum_{n=0}^{\infty}\int_0^T \{B_n \cos(p\omega_0 t)\sin(n\omega_0 t)\}dt} \qquad \textbf{(3)}$$

Using the trigonometric identity 2cosAsinB=sin (A+B)-sin (A-B), and sin($2\pi t/T$) = 0, note that the second term in the equation (3) is equal to zero, i.e.,

$$\sum_{n=0}^{\infty}\int_0^T \{B_n \cos(p\omega_0 t)\sin(n\omega_0 t)\}dt = \frac{B_n}{2}\int_0^T \{\sin(p+n)\omega_0 t - \sin(p-n)\omega_0 t\}dt$$

$$= \frac{B_n}{2}\int_0^T \sin\left[\frac{(p+n)2\pi t}{T}\right]dt - \frac{B_n}{2}\int_0^T \sin\left[\frac{(p-n)2\pi t}{T}\right]dt = 0 \qquad \textbf{(4)}$$

Eq: (4) is true for all positive integers of p and n.

Using trigonometric identity 2cosAcosB=cos (A+B)+ cos (A-B), we find that the first term of Eq: (3) is only equal to zero when p ≠n, i.e.,

$$\int_0^T \{A_n \cos(p\omega_0 t)\cos(n\omega_0 t)\}dt = \frac{A_n}{2}\int_0^T \{\cos(p+n)\omega_0 t + \cos(p-n)\omega_0 t\}dt = 0 \qquad \textbf{(5)}$$

If p=n, then

$$\int_0^T \{A_n \cos(p\omega_0 t)\cos(n\omega_0 t)\}dt = A_n \int_0^T \cos^2(n\omega_0 t)dt$$

$$= \frac{A_n}{2}\int_0^T (1+\cos 2n\omega_0 t)dt = \frac{A_n}{2}\int_0^T 1.dt = \frac{A_n}{2} t\Big|_0^T = \frac{A_n T}{2} \qquad \textbf{(6)}$$

Therefore, using Eq: (6), (5), (4), and (3), we note that:

$$\int_0^T \{\cos(p\omega_0 t)x(t)\}dt = \frac{A_n T}{2},$$

and therefore, since p=n, $$A_n = \frac{2}{T}\int_0^T \{\cos(n\omega_0 t)x(t)\}dt \qquad \textbf{(7)}$$

Fourier Series Representation

By multiplying Eq: (3) by sin ($p\omega_0 t$) and using a similar set of simplifications we can show that:

$$B_n = \frac{2}{T}\int_0^T \{x(t)\sin(n\omega_0 t)\}dt \qquad (8)$$

Hence, the three key equations for calculating the Fourier Series of a periodic signal with fundamental period T are:

$$x(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$A_n = \frac{2}{T}\int_0^T \{x(t)\cos(n\omega_0 t)\}dt$$

$$B_n = \frac{2}{T}\int_0^T \{x(t)\sin(n\omega_0 t)\}dt$$

Complex Fourier Series

From Euler's theorem, note that:

$$e^{j\omega} = \cos(\omega) + j\sin(\omega) \qquad \cos(\omega) = \left(e^{j\omega} + e^{-j\omega}\right)/2 \qquad \sin(\omega) = \left(e^{j\omega} - e^{-j\omega}\right)/2j$$

Substituting these values in Eq: (1), and rearranging gives:

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right)\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left[A_n\left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + B_n\left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left[\left(\frac{A_n}{2} + \frac{B_n}{2j}\right)e^{jn\omega_0 t} + \left(\frac{A_n}{2} - \frac{B_n}{2j}\right)e^{-jn\omega_0 t}\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n - jB_n}{2}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{A_n + jB_n}{2}\right)e^{-jn\omega_0 t} \qquad (9)$$

For the second summation term, if the sign of the complex sinusoid is negated and the summation limits are reversed, then we can rewrite as:

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n - jB_n}{2}\right)e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1}\left(\frac{A_n + jB_n}{2}\right)e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad (10)$$

Complex Fourier Series

where Cn in terms of the Fourier series coefficients of Eq. 7 and 8 gives:

$$
\begin{aligned}
&C_0 = A_0 \\
&C_n = (A_n - jB_n)/2 \qquad \text{for n > 0} \\
&C_n = (A_n + jB_n)/2 \qquad \text{for n < 0}
\end{aligned}
\qquad \textbf{(11)}
$$

From Eq. 11 note that for n > 0,

$$C_n = \frac{A_n - jB_n}{2} = \frac{1}{T}\int_0^T x(t)\cos(n\omega_0 t)dt - j\frac{1}{T}\int_0^T x(t)\sin(n\omega_0 t)dt$$

$$= \frac{1}{T}\int_0^T x(t)[\cos(n\omega_0 t) - j\sin(n\omega_0 t)]dt = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt \qquad \textbf{(12)}$$

For n < 0, it is clear from Eq. 11 that, $C_n = C^*_{-n}$, where '*' denotes complex conjugate. Therefore, the two important equation for complex exponential Fourier series are

$$\boxed{\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \\ C_n &= \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt \end{aligned}}$$

Example

The ease of working with complex exponentials can be illustrated by this simple example.

Example 1: Simplify the following equations in to a sum of sine waves:

$$\sin(\omega_1 t)\sin(\omega_2 t)$$

This requires the recollection (or rederivation!) of trigonometric identities to yield:

$$\sin(\omega_1 t)\sin(\omega_2 t) = \frac{1}{2}\cos(\omega_1 - \omega_2)t + \frac{1}{2}\cos(\omega_1 + \omega_2)t$$

However, it is relatively easier to simplify the following expression to a sum of complex exponentials:

$$e^{j\omega_1 t} e^{j\omega_2 t} = e^{j(\omega_1 + \omega_2)t}$$

Although, seemingly a simple comment this is the basis of using complex exponentials rather than sines and cosines; they make the maths easier. Of course, in situations where the signal being analysed is complex, then the complex Fourier series *must* be used!

Fourier Transform

- The Fourier Series allows a *periodic* signal to be broken down into a sum of sin and cos components.
- However, most practical signals are aperiodic!
- Therefore, the Fourier Transform was derived in order to analyse the frequency content of *aperiodic* signals.

Discrete Fourier Transform

The DT Fourier Transform (DTFT) of a finite energy discrete time signal x[n] is defined as:

$$X(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad \omega \in [-\pi, \pi]$$

$X(\omega)$ may be regarded as a decomposition of x[n] into its *frequency components*.

- It is not difficult to verify that $X(\omega)$ is periodic with frequency 2π.

The Inverse Fourier Transform of $X(\omega)$ may be defined as:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

Discrete Fourier Transform

- Notation: $x[n] \leftrightarrow X(\omega)$, $x[n]=F^{-1}(X(\omega))$, $X(\omega)=F^{-1}(x[n])$
- Signal has a transform if it satisfies Dirichlet conditions.
- $X(\omega)$ is called the spectrum of x[n]:

$$X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)} \Rightarrow \begin{cases} |X(\omega)| = & \text{magnitude spectrum,} \\ \angle X(\omega) = & \text{phase spectrum,} \end{cases}$$

The magnitude spectrum is often expressed in deceibels (dB)

- DTFT describes the frequency content of x[n]
- For real signals
  - $|X(\omega)|=|X(-\omega)| \rightarrow$ Even function, and
  - phase $\angle X(-\omega) = -\angle X(\omega) \rightarrow$ Odd function.

$$\boxed{\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \\ x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \end{aligned}}$$

Energy Density Spectrum of Aperiodic Signals

Energy of a discrete time signal x[n] is defined as:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Let us now express the energy Ex in terms of the spectral characteristic X(w). First we have

$$E_x = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)e^{-j\omega n}d\omega\right]$$

If we interchange the order of integration and summation in the above equation, we obtain

$$E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right]d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Therefore, the energy relation between x[n] and X(w) is

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega}$$

Parseval's relation for DT Aperiodic signals

Energy Density Spectrum of Aperiodic Signals

- The spectrum is, in general, a complex valued function of frequency.
- The quantity $S_{xx}(w)=|X(w)|^2$ represents the distribution of energy as a function of frequency and it is called Energy Density Spectrum of x(n).
- $S_{xx}(w)$ does not contain any phase information.

Energy Density Spectrum of Aperiodic Signals

Example - 1: Determine DTFT and sketch the energy density spectrum $S_{xx}(w)$ of the sequence:

$$x[n]=\alpha^n u[n] \qquad |\alpha|<1$$

Solution- 1:

$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(\omega)=\sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}=\sum_{n=0}^{\infty}\left(\alpha e^{-j\omega}\right)^n$$

Using the geometric sequence, provided $|\alpha|<1$, this sum is:

$$\boxed{X(\omega)=\frac{1}{1-\alpha e^{-j\omega}}}$$

Energy Density Spectrum of Aperiodic Signals

Energy Density Spectrum is given by

$$S_{xx}(\omega) = |X(\omega)|^2 = X(\omega)X^*(\omega)$$

$$S_{xx}(\omega) = \frac{1}{(1 - ae^{-j\omega})} \frac{1}{(1 - ae^{j\omega})}$$

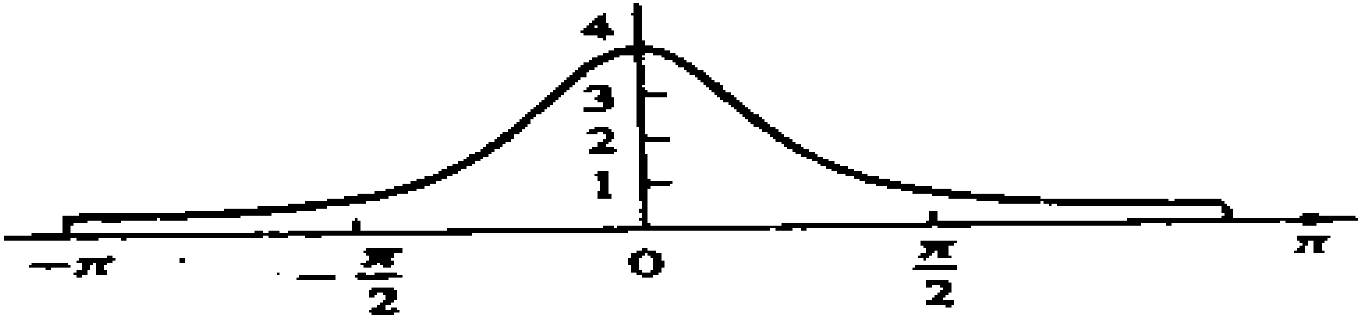
$$S_{xx}(\omega) = \frac{1}{1 - a(e^{jw} + e^{-jw}) + a^2}$$

$$S_{xx}(\omega) = \frac{1}{1 - 2a\cos\omega + a^2}$$

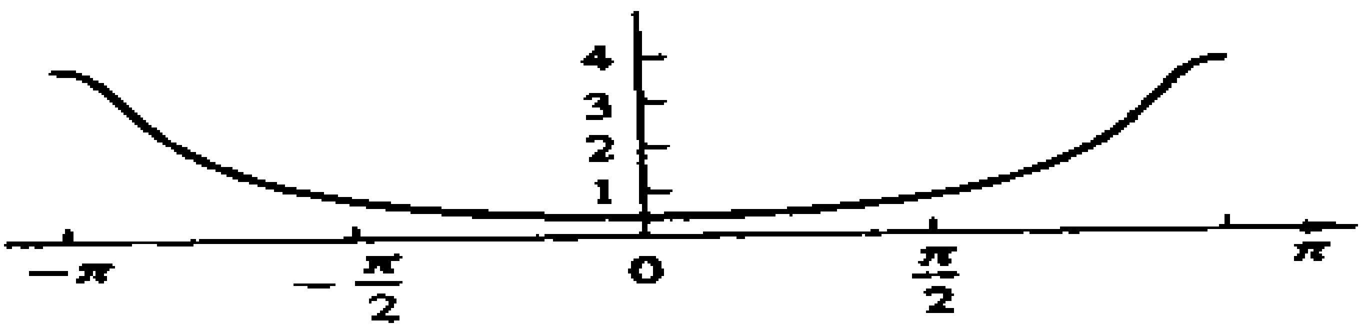
Figure on next slide shows x(n) and its corresponding spectrum for a=0.5 & a=-0.5

Energy Density Spectrum of Aperiodic Signals

$$S_{xx}(\omega) = \frac{1}{1 - 2a\cos\omega + a^2}, \quad a = 0.5$$



$$S_{xx}(\omega) = \frac{1}{1 - 2a\cos\omega + a^2}, \quad a = -0.5$$



Energy Density Spectrum of Aperiodic Signals

Example – 2: Determine the Fourier Transform and the energy density spectrum of the sequence

$$x[n] = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & otherwise \end{cases}$$

Solution – 2:

$$X(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{0}^{L-1} A e^{-j\omega n} = A \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

The magnitude of x[n] is

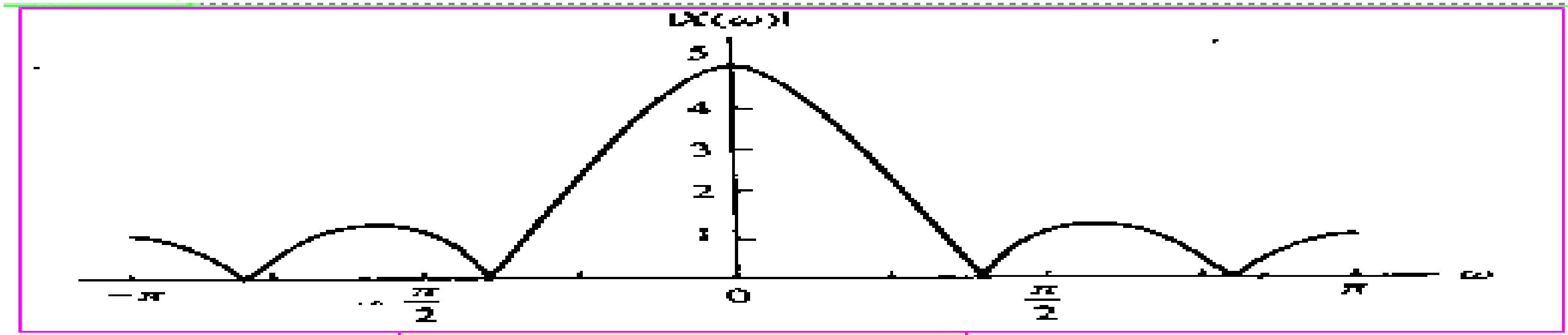
$$|X(\omega)| = \begin{cases} |A| L, & \omega = 0 \\ |A| \left| \dfrac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, & \text{otherwise} \end{cases}$$

and the phase spectrum is

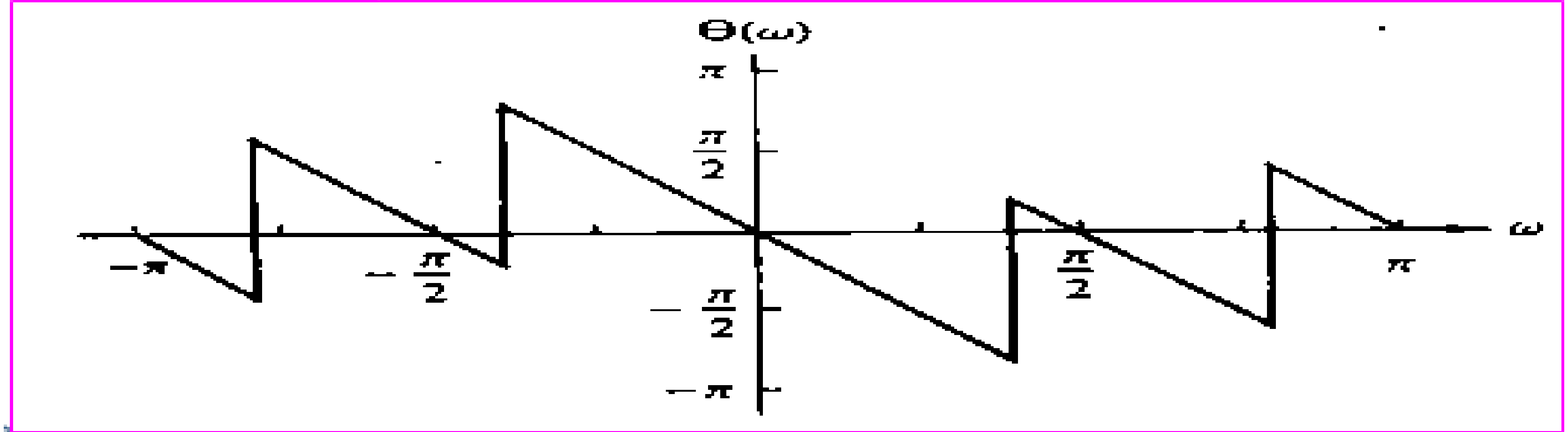
$$\angle X(\omega) = \angle A - \angle \frac{\omega}{2}(L-1) + \angle \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

The signal x[n] magnitude and phase is plotted on the next slide.

Energy Density Spectrum of Aperiodic Signals



Magnitude response



Phase response

Some Common DTFT

Sequence	Discrete-Time Fourier Transform
$\delta(n)$	1
$\delta(n-n_0)$	$e^{-jn_0\omega}$
1	$2\pi\delta(\omega)$
$e^{jn\omega_0}$	$2\pi\delta(\omega-\omega_0)$
$a^n u(n), \quad \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\omega}}$
$-a^n u(-n-1), \quad \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u(n), \quad \lvert a\rvert < 1$	$\dfrac{1}{(1-ae^{-j\omega})^2}$
$\cos n\omega_0$	$\pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0)$

Properties of DTFT

- A FT for Aperiodic finite energy DT signals described possesses a number of properties that are very useful in reducing the complexity of frequency analysis problems in many practical applications.
- For convenience, we adopt the notations

$$x[n] \xleftrightarrow{F} X(\omega)$$

$$x[n]=F^{-1}\{X(\omega)\}$$

$$X(\omega)=F^{-1}(x[n])$$

Properties of DTFT

- **Symmetry:**
  - Real and even $x(n) \rightarrow$ Real and Even $X(\omega)$
  - Real and odd $x(n) \rightarrow$ Imaginary and odd $X(\omega)$
  - Imaginary and odd $x(n) \rightarrow$ Real and odd $X(\omega)$
  - Imaginary and even $x(n) \rightarrow$ Imaginary and even $X(\omega)$
- **Linearity:**
  - If $x_1[n] \xleftrightarrow{F} X_1(\omega)$

    $x_2[n] \xleftrightarrow{F} X_2(\omega)$

$$\boxed{a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{DTFT} a_1 X_1(\omega) + a_2 X_2(\omega)}$$

Properties of DTFT

Example – 3: Determine the DTFT of the signal $x[n] = a^{|n|}$

Solution – 3: First, we observe that $x[n]$ can be expressed as: $x[n]=x_1[n]+ x_2[n]$ (linearity prop:)

Where $$x_1[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{and} \quad x_2[n] = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

Now, $$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

$$= 1 + ae^{-j\omega} + \left(ae^{-j\omega}\right)^2 + \left(ae^{-j\omega}\right)^3 + \ldots = \frac{1}{1 - ae^{-j\omega}}$$

and, $$X_2(\omega) = \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{-1} \left(ae^{j\omega}\right)^{-n}$$

$$= \sum_{k=1}^{\infty} \left(ae^{j\omega}\right)^k = ae^{j\omega} + (ae^{j\omega})^2 + \ldots = \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

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Properties of DTFT

- Time shifting:
  - If $x_1[n] \xleftrightarrow{F} X_1(\omega)$
  - Then $\boxed{x[n-k] \xleftrightarrow{DTFT} e^{-j\omega k} X(\omega)}$

Proof: Taking FT of x[n-k]

$$F\left[x[n-k]\right] = \sum_{n=-\infty}^{\infty} x[n-k] e^{-j\omega n}$$

Let n – k = m or n = m+k

$$\therefore F\left[x[n-k]\right] = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+k)} = e^{-j\omega k} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} = e^{-j\omega k} X(\omega)$$

Similarly for x[n+k], $F\{x[n+k]\} = e^{jwk} X(w)$

Properties of DTFT

- **Time reversal:**
  - If $x[n] \xleftrightarrow{F} X(\omega)$
  - Then $\boxed{x[-n] \xleftrightarrow{DTFT} X(-\omega)}$

Proof: Let m = -n

$$F[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n}$$

$$F[x[-n]] = \sum_{m=\infty}^{-\infty} x[m] e^{j\omega(-m)} = \sum_{m=-\infty}^{\infty} x[m] e^{-j(-\omega m)} = X(-\omega)$$

Properties of DTFT

- Convolution:
  - If

$$x_1[n] \xleftrightarrow{F} X_1(\omega)$$

$$x_2[n] \xleftrightarrow{F} X_2(\omega)$$

$$\boxed{x[n] = x_1[n] * x_2[n] \xleftrightarrow{DTFT} X(\omega) = X_1(\omega)X_2(\omega)}$$

This theorem is one of the most powerful tool. If we convolve 2 signals in time Domain, then this is equal to multiplying Their spectra in the freq: domain.

Proof: Recall convo: formula

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

Multiply both sides of this eq: by e-jwn and sum over all n, we get

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x[k]y[n-k] \right] e^{-j\omega n}$$

Interchanging the order of summation and making a substitution n – k = m,

$$X(\omega) = \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{m=-\infty}^{\infty} x_2[m] \right] e^{-j\omega(m+k)} = \left[\sum_{k=-\infty}^{\infty} x_1[k]e^{-j\omega k} \right] \left[\sum_{m=-\infty}^{\infty} x_2[m]e^{-j\omega m} \right]$$

$$X(\omega) = X_1(\omega)X_2(\omega)$$

Example – 4: Determine the convolution of the sequences $x_1[n] = x_2[n] = [1\ \ \underline{1}\ \ 1]$

Solution – 4:

$$X_1(\omega) = X_2(\omega) = \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n} = \sum_{n=-1}^{1} x_1[n]e^{-j\omega n}$$

$$= \left[x_1[-1]e^{j\omega} + x_1[0]e^{0} + x_2[1]e^{-j\omega}\right] = \left[e^{j\omega} + 1 + e^{-j\omega}\right]$$

$$= 1 + 2\cos\omega$$

Therefore,

$$X(\omega) = X_1(\omega)X_2(\omega) = (1+2\cos\omega)^2 = 1 + 4\cos\omega + 4\cos^2\omega$$

$$= 1 + 4\cos\omega + \frac{4}{2}(1+\cos 2\omega) = 3 + 4\cos\omega + 2\cos 2\omega$$

$$= 3 + 2\left(e^{j\omega} + e^{-j\omega}\right) + \left(e^{j2\omega} + e^{-j2\omega}\right)$$

$$X(\omega) = X_1(\omega)X_2(\omega) = e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega}$$

Hence the convolution of $x_1[n]$ and $x_2[n]$ is $x[n] = [1\ \ 2\ \ \underline{3}\ \ 2\ \ 1]$

Properties of DTFT

- **Correlation:**
  - If
    $$x_1[n] \xleftrightarrow{F} X_1(\omega)$$
    $$x_2[n] \xleftrightarrow{F} X_2(\omega)$$
  - Then
    $$\boxed{r_{x_1x_2} \xleftrightarrow{DTFT} S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)}$$
- **The Wiener- Khintchine Theorem:**
  - Let x(n) be a real signal, then
    $$\boxed{r_{xx}(l) \xleftrightarrow{DTFT} S_{xx}(\omega)}$$
  - That is, *the DTFT of autocorrelation function is equal to its energy density function*. This is a special case.
  - Autocorrelation sequence of a signal & its energy spectral density contain the same info: about the signal.

Properties of DTFT

Proof: The autocorrelation of x[n] is defined as

$$r_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[k-n]$$

Now

$$F[r_{xx}[n]] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x[k]x[k-n]\right]e^{-j\omega n}$$

Re-arranging the order of summations and making Substitution m= k-n,

$$F[r_{xx}[n]] = \sum_{k=-\infty}^{\infty} x[k]\left[\sum_{m=\infty}^{-\infty} x[m]\right]e^{-j\omega(k-m)} = \left[\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\right]\left[\sum_{m=-\infty}^{\infty} x[m]e^{j(-\omega m)}\right]$$

$$= X(\omega)X(-\omega) = |X(\omega)|^2 = S_{xx}(\omega)$$

Properties of DTFT

- Frequency shifting:
  - If $x[n] \xleftrightarrow{F} X(\omega)$
  - Then $\boxed{e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(\omega - \omega_0)}$

According to this property, multiplication of a sequence x(n) by e^{jw0n}, is equivalent to a frequency translation of the spectrum X(w) by w_0

Proof:

$$F\left[x[n]e^{j\omega_0 n}\right] = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega_0 n}e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} = X(\omega-\omega_0)$$

Properties of DTFT

Modulation theorem:

- If $x[n] \xleftrightarrow{F} X(\omega)$
- Then $\boxed{x[n]\cos\omega_0 n \xleftrightarrow{DTFT} \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]}$

Parseval's Theorem:

- If $x_1[n] \xleftrightarrow{F} X_1(\omega)$

  $x_2[n] \xleftrightarrow{F} X_2(\omega)$
- Then

$$\boxed{\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega}$$

Properties of DTFT

Proof:

$$R.H.S. = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n}\right]X_2^*(\omega)d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)e^{-j\omega n}d\omega = \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = L.H.S$$

In the special case where $x_1[n] = x_2[n] = x[n]$, the Parseval's Theorem reduces to:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

We observe that the LHS of the above equation is energy Ex of the Signal and the R.H.S is equal to the energy density spectrum. Thus we can re-write the above equation as:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)d\omega$$

Properties of DTFT

- Multiplication of two sequences (Windowing theorem):
  - If $x_1[n] \xleftrightarrow{F} X_1(\omega)$

    $x_2[n] \xleftrightarrow{F} X_2(\omega)$
  - Then $\boxed{x_3 \equiv x_1[n]x_2[n] \xleftrightarrow{DTFT} X_3(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\lambda)Y(\omega-\lambda)d\lambda}$

This theorem states that: The multiplication of two time domain sequences is equivalent to the convolution of their Fourier transforms.

Proof:

$$F\left[x_1[n]x_2[n]\right] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\lambda)e^{j\lambda n}d\lambda\right]x_2[n]e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x_1(\lambda)d\lambda\left[\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\lambda)n}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} x_1(\lambda)x_2(\omega-\lambda)d\lambda$$

Differentiation in the Frequency domain:

- If $x[n] \xleftrightarrow{F} X(\omega)$
- Then $\boxed{nx[n] \xleftrightarrow{DTFT} j\dfrac{dX(\omega)}{d\omega}}$

Proof:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right] = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{d\omega}e^{-j\omega n}$$

$$\frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}(-jn)$$

$$j\frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$$

Properties of DTFT

- Linearity: $a_1x_1[n]+a_2x_2[n] \xleftrightarrow{DTFT} a_1X_1(\omega)+a_2X_2(\omega)$
- Time shifting: $x[n-k] \xleftrightarrow{DTFT} e^{-j\omega k}X(\omega)$
- Conjugate: $x^*[n] \xleftrightarrow{DTFT} X^*(e^{-j\omega}) = X^*(\omega)$
- Time reversal: $x[-n] \xleftrightarrow{DTFT} X(-\omega)$
- Frequency shifting: $e^{j\omega_0 n}x[n] \xleftrightarrow{DTFT} X(\omega-\omega_0)$
- Differentiation: $nx[n] \xleftrightarrow{DTFT} j\dfrac{dX(\omega)}{d\omega}$
- Convolution: $x[n]=x_1[n]*x_2[n] \xleftrightarrow{DTFT} X(\omega)=X_1(\omega)X_2(\omega)$
- Correlation: $r_{x_1x_2} \xleftrightarrow{DTFT} S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
- Wiener khinchine: $r_{xx}(l) \xleftrightarrow{DTFT} S_{xx}(\omega)$
- Multiplication: $x_3 \equiv x_1[n]x_2[n] \xleftrightarrow{DTFT} X_3(\omega) = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(\lambda)Y(\omega-\lambda)d\lambda$
- Parseval's Theorem: $\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] \xleftrightarrow{DTFT} \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$
- Modulation Theorem: $x[n]\cos\omega_0 n \xleftrightarrow{DTFT} \dfrac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$

Properties of DTFT

Tutorial:

1. Find the DTFT of $y[n]=-\alpha^n u(-n-1)$, provided $|\alpha|>1$
2. Prove correlation property of DTFT.
3. Prove modulation theorem of DTFT.
4. Prove differentiation property of DTFT.
5. An LTI system is characterized by its impulse response $h[n] = (1/2)^n u[n]$. Determine the spectrum and the energy density spectrum of the output signal when the system is excited by the signal $x[n] = (1/4)^n u[n]$.

Discrete Fourier Transform

- Recall the definition of DTFT:

$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \longrightarrow \quad (1)$$

- While the DTFT is useful from a theoretical point of view, its numerical evaluation poses difficulties:
  - The summation over n is infinite
  - The variable ω is continuous
- In many situations of interest, it is either not possible, or not necessary to implement the infinite summation in (1).
  - Only the signal samples of $x[n]$ from n=0 to N-1 are available;
  - The signal is known to zero outside this range; or
  - The signal is periodic with period N.
- In all these cases, we would like to analyze the frequency content of signal $x[n]$ based only on the finite set of samples $x[0]$, $x[1]$, … , $x[N-1]$.
- We would also like a frequency domain representation of these samples in which the frequency variable only take a finite set of values, say ω_k for $k=0, 1, \ldots, N-1$.
- The Discrete Fourier Transform (DFT) fulfils these needs. It can be seen as an approximation to the DTFT.

Definition: Discrete Fourier Transform

The N-point DFT is a transformation that maps DT signal samples $\{x[0], \ldots ,x[N\text{-}1]\}$ into a periodic sequence $x[k]$, defined by

$$x[k] = DFT_N\{x[n]\} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \qquad k \in Z$$

Remarks:

- Only the samples $x[0], \ldots ,x[N\text{-}1]$, are used in computation.
- The N-point DFT is periodic, with period N: $x[k+N]=x[k]$.
  Thus it is sufficient to specify $x[k]$ for $k=0,1, \ldots , N\text{-}1$.

Inverse DFT (IDFT)

Definition: Inverse DFT

The N-point IDFT of the samples $x[0], \ldots, x[N-1]$ is defined as the periodic sequence $x[k]$, defined by:

$$\tilde{x}[n] = IDFT_N\{X[k]\} = \frac{1}{N}\sum_{n=0}^{N-1} x[k]e^{j2\pi kn/N}, \qquad k \in Z$$

Remarks:

- In general, $\tilde{x}[n] \neq x[n]$ for all n є Z
- Only the samples, $x[0], \ldots, x[N-1]$, are used in the computation.
- The N-point DFT is periodic, with period N: $\tilde{x}[n+N] = x[k]$

IDFT Theorem

IDFT Theorem:

If $X[k]$ is the N-point DFT of $\{x[0], \ldots x[N-1]\}$, then

$$\tilde{x}[n] = x[n], \qquad n = 0,1,\ldots,N-1 \text{ only.}$$

Remarks:

- Theorem states that $\tilde{x}[n] = x[n]$ for $n = 0,1, \ldots, N$-1 only.
- In general, the values of $x[n]$ for $n < 0$ and for $n \geq N$ cannot be recovered from the DFT samples $X[k]$. This is understandable since these sample values are not used when computing $X[k]$.
- However, there are two important cases when the complete signal $x[n]$ can be recovered from the DFT samples $X[k]$ (k=0,1,..., N-1)
  - $x[n]$ is periodic with period N.
  - $x[n]$ known to be zero for $n < 0$ and for $n \geq N$.

$$\boxed{\begin{aligned} x[k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \\ \tilde{x}[n] &= \frac{1}{N} \sum_{n=0}^{N-1} x[k] e^{j2\pi kn/N} \end{aligned}}$$

Example

Example: Prove that DFT is periodic with period N.

Proof: we know that, DFT is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk2\pi n/N}$$

Therefore,

$$X[k+N] = \sum_{n=0}^{N-1} x[n] e^{-j(k+N)2\pi n/N} = \sum_{n=0}^{N-1} x[n] e^{-jk2\pi n/N} e^{-j2\pi n}$$

Sine $e^{-j2\pi n} = 1$,

$\therefore X[k+N] = \sum_{n=0}^{N-1} x[n] e^{-jk2\pi n/N} = x[k]$ hence, proved.

Example

Example: Find the DFT of x[n] = [1 0 0 1]

Solution: $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk2\pi n/N} = \sum_{n=0}^{3} x[n] e^{-jk2\pi n/4} = \sum_{n=0}^{3} x[n] e^{-jk\pi n/2}$$

Now, $$X[0] = \sum_{n=0}^{3} x[n] = x[0] + x[1] + x[2] + x[3] = 1 + 0 + 0 + 1 = 2$$

$$X[1] = \sum_{n=0}^{3} x[n] e^{-jk\pi n/2} = x[0] + 0 + 0 + x[3] e^{-j3\pi/2}$$

$$= 1 + 1 . e^{-j3\pi/2} = 1 + \cos(\tfrac{3\pi}{2}) - j \sin(\tfrac{3\pi}{2}) = 1 + j$$

$$X[2] = \sum_{n=0}^{3} x[n] e^{-j\pi n} = x[0] + x[3] e^{-j3\pi n}$$

$$= 1 + 1 .[\cos(3\pi n) - j \sin(3\pi n)] = 0$$

$$X[3] = \sum_{n=0}^{3} x[n] e^{-j3\pi n/2} = x[0] + x[3] e^{-j9\pi/2} = 1 - j$$

Example

Example: Find the IDFT of the sequence y[n]=[2 1+i 0 1-i]

Solution: $x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{jk2\pi n/N}$

$$x[0]=\frac{1}{4}\sum_{k=0}^{N-1}X[k]=\frac{1}{4}[2+(1+i)+0+(1-i)]=1$$

$$x[1]=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{jk2\pi/4}=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{jk\pi/2}=\frac{1}{4}\left[2+(1+i)e^{j\pi/2}+0.e^{j\pi}+(1-i)e^{j3\pi/2}\right]=0$$

$$x[2]=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{jk4\pi/4}=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{jk\pi}=\frac{1}{4}\left[2+(1+i)e^{j\pi}+0.e^{j2\pi}+(1-i)e^{j3\pi}\right]=0$$

$$x[3]=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{jk6\pi/4}=\frac{1}{4}\sum_{k=0}^{3}X[k]e^{jk3\pi/2}=\frac{1}{4}\left[2+(1+i)e^{j3\pi/2}+0.e^{j3\pi}+(1-i)e^{j9\pi/2}\right]=1$$

Periodicity:

The N-point DFT is periodic, with period N: $\tilde{x}[n+N]=x[k]$

Linearity:

If $x[n]$ and $y[n]$ have N-point DFTs $X(k)$ and $Y(k)$, respectively,

$$ax[n]+by[n] \xleftrightarrow{DFT} aX(k)+bY(k)$$

In using this property, it is important to ensure that the DFTs are the same length. If $x[n]$ and $y[n]$ have different lengths, then shorter sequence must be *padded* with zeros in order to make it the same length as the longer sequence.

Symmetry:

If $x[n]$ is real-valued, $X(k)$ is *conjugate symmetric,*

$$X(k)=X^*((-k))=X^*((N-k))_N$$

and if $x[n]$ is imaginary, $X(k)$ is *conjugate antisymmetric,*

$$X(k)=-X^*((-k))=-X^*((N-k))_N$$

Example

Example: A finite duration sequence of length L is given by

$$x[n] = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the N point DFT of this sequence for N≥L.

Solution: The DTFT of the sequence was calculated as

$$X(w) = \frac{\sin(wL/2)}{\sin(w/2)} e^{-jw(L-1)/2}$$

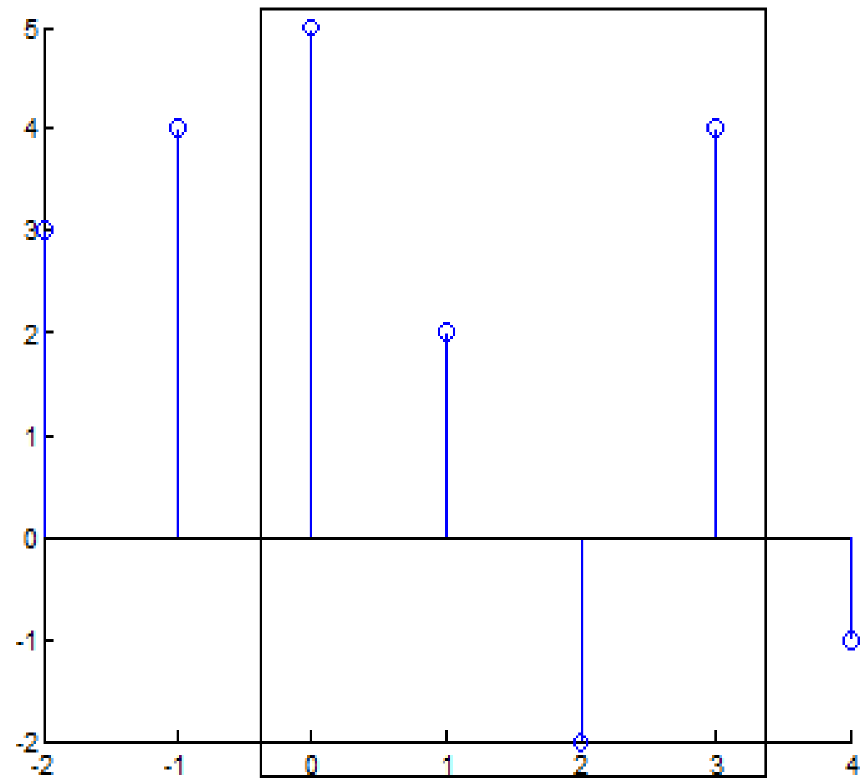
The N point DFT is simply X(w) evaluated at the set of N equally spaced frequencies $w_k = 2\pi k/N$, $k = 0,1,\ldots,N-1$. Hence

$$X(k) = \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-jwk(L-1)/N}$$

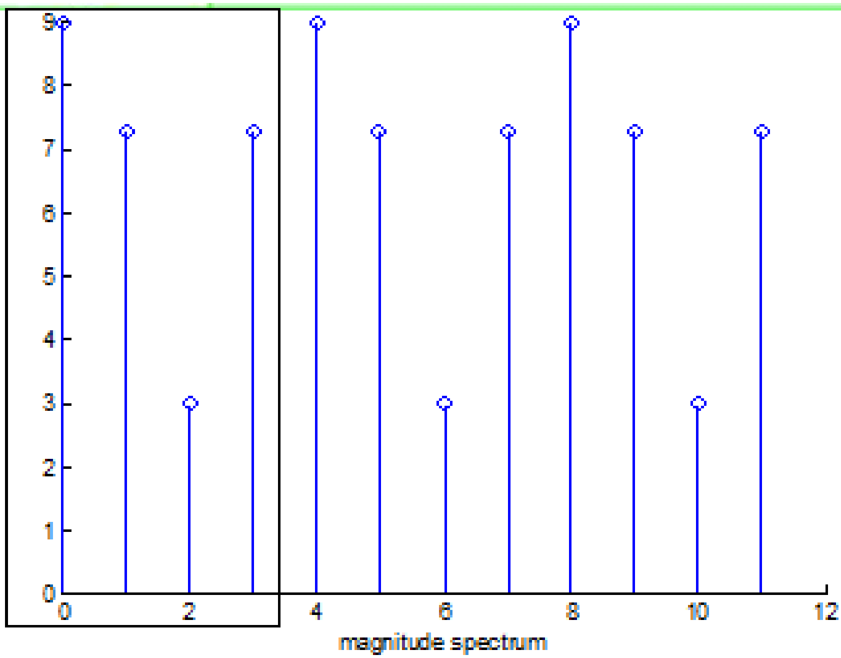
Example

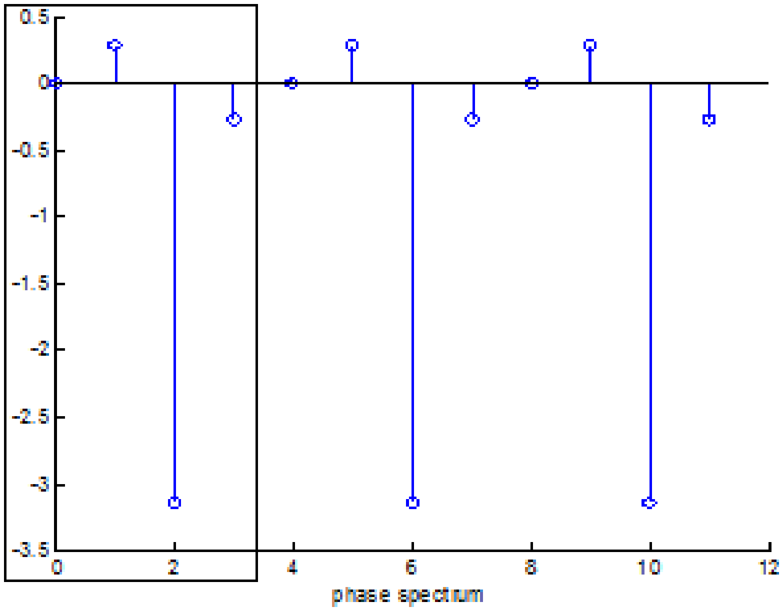
- Example: Find DFT magnitude and phase spectra for the samples of the signal selected in figure. Also verify that IDFT reproduces these samples.
- Solution:

K	X[k]	|X[k]|	<θ radians
0	9	9	0
1	7+2j	7.2801	0.2782
2	-3	3	-3.1416
9	7-2j	7.2801	-0.2782



Example

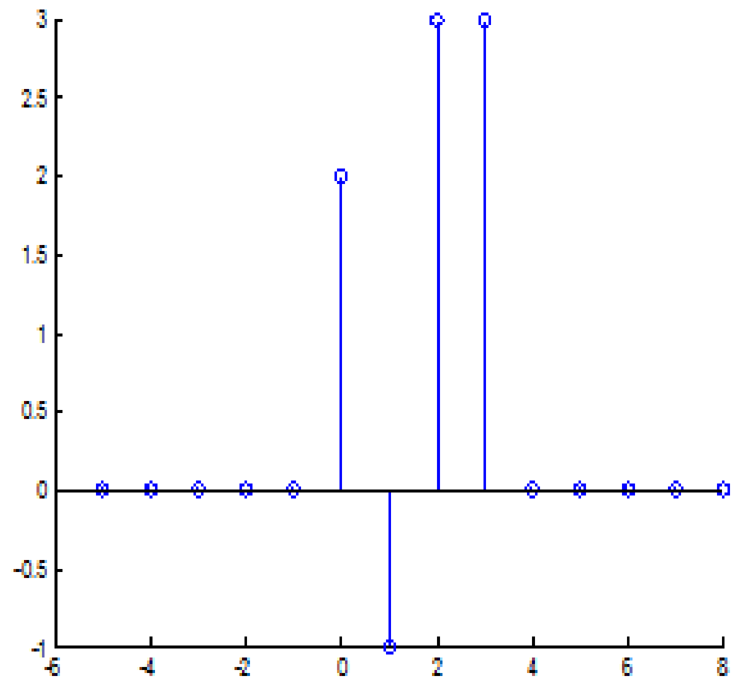




Example

Tutorial:

1. Find the DFT of x[n]=[2 1 0 2]
2. Find the IDFT of X[k]=[1+i 0 1 1-i]
3. Find the DFT of the 4-point sequence x[n] = [0 1 2 3]
4. Find the 4 point IDFT of the sequence [6,-2+2j,-2,-2-2j].
5. Find magnitude spectrum using both DTFT and DFT for the signal shown here.



Fast Fourier Transform (FFT)

Recap:

- Let $x[n]$ be a discrete-time signal defined for $0 \le n \le N\text{-}1$.
- The DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0,1,\ldots, N-1 \qquad (1)$$

$$W_N = e^{-j2\pi/N} = \cos(2\pi/N) + j\sin(2\pi/N)$$

- Notes:
  - Note that the direct computation of DFT requires N^2 computations.
  - The same is true for IDFT
  - The FFT only requires $N\log_2 N$ calculations.
  - The computational saving achieved by FFT is therefore a factor of $N/\log_2 N$. When N is large this saving can be significant.
  - The following table compares the number of calculations required for different values of N for the DFT and FFT:

N	DFT	FFT
32	1024	160
1024	1048576	10240
32768	~ 1 x 10^9	~ 0.5 X 10^6

Fast Fourier Transform (FFT)

- <u>What is FFT</u>
  - FFT stands for Fast Fourier Transform
  - FFT is a method of computing the Discrete Fourier Transform (DFT) that exploits the redundancy in the general DFT equation given in (1).
  - The FFT is not a new transform; it refers to a family of efficient algorithms for computing the DFT.
  - Typically, FFT requires $N\log_2 N$ while DFT requires N^2.
- <u>Basic Principle</u>
  - The FFT relies on the concept of divide and conquer
  - It is obtained by breaking the DFT of size N into a cascade of smaller size DFTs.
  - To achieve this:
    - N must be a composite number
    - The properties of W_N must be exploited, e.g.;

$$W_N^k = W_N^{k+N} \quad (2)$$

$$W_N^{Lk} = W_{N/L}^k \quad (3)$$

Example

Example: We can highlight the existence of redundant computations in the DFT by inspecting Eq. (1). Using the DFT algorithm to calculate the first four components of the DFT of a signal with only 8 samples requires the following computations:

$$\begin{aligned}
X[0] &= x[0]+x[1]+x[2]+x[3]+x[4]+x[5]+x[6]+x[7] \\
X[1] &= x[0]+x[1]W_8^{-1}+\boxed{x[2]W_8^{-2}}+x[3]W_8^{-3}+x[4]W_8^{-4}+x[5]W_8^{-5}+x[6]W_8^{-6}+x[7]W_8^{-7} \\
X[2] &= x[0]+x[1]W_8^{-2}+x[2]W_8^{-4}+x[3]W_8^{-6}+x[4]W_8^{-8}+x[5]W_8^{-10}+x[6]W_8^{-12}+x[7]W_8^{-14} \\
X[3] &= x[0]+x[1]W_8^{-3}+\boxed{x[2]W_8^{-6}}+x[3]W_8^{-9}+x[4]W_8^{-12}+x[5]W_8^{-15}+x[6]W_8^{-18}+x[7]W_8^{-21}
\end{aligned} \quad \textbf{(4)}$$

However note that there is redundant (repeated) terms in Eq. (4). For e.g., consider 3rd term in 2nd line of Eq. (4).

$$x[2]W_8^{-2} = x[2]e^{j2\pi\left(\frac{-2}{8}\right)} = x[2]e^{\frac{-j\pi}{2}}$$

Now, consider the computation of third term in the fourth line of Eq. (4):

$$x[2]W_8^{-6} = x[2]e^{j2\pi\left(\frac{-6}{8}\right)} = x[2]e^{\frac{-j3\pi}{2}} = x[2]e^{-j\pi}e^{\frac{-j\pi}{2}} = -x[2]e^{\frac{-j\pi}{2}}$$

Therefore we can save one multiply operation by noting that $x[2]W_8^{-6} = -x[2]W_8^{-2}$

In fact because of the periodicity of $x[k]W_N^{nk}$ every term in the fourth line of Eq. (4) is available from the computed terms in the second line of the equation.

Example

More generally, we can show that the terms in the second line of Eq. (4) are:

$$x[k]W_8^{-k} = x[k]e^{\frac{-j2\pi k}{2}} = x[k]e^{\frac{-j\pi k}{2}}$$

and for the terms in fourth line of Eq. (4):

$$x[k]W_8^{-3k} = x[k]^{-j\frac{6\pi k}{2}} = x[k]e^{-j\frac{3\pi k}{2}} = x[k]e^{-j\left(\frac{\pi}{2}+\frac{\pi}{4}\right)k}$$

$$= x[k]e^{-j\frac{\pi k}{2}}e^{-j\frac{\pi k}{4}} = x[k](-j)^k e^{-j\frac{\pi k}{4}} = (-j)^k x[k]W_8^{-k}$$

This exploitation of the computational redundancy is the basis of FFT which allows the same results as the DFT to be computed, but with less computations.

Different Types of FFT

- There are several FFT algorithms sometimes grouped via the names Cooley- Tukey, prime factor, decimation in time, decimation in frequency, radix-2 and so on. The bottom line for all FFT algorithms is, however, that they remove redundancy from the direct DFT computational algorithm of Eq. (1).

Notable Examples of FFT Algorithms:

- $N = 2^v \rightarrow$ Radix – 2 FFTs. These are the most commonly used algorithms. Even then, there are many variations:
  - Decimation in Time (DIT)
  - Decimation in Frequency (DIF)

Radix-2 are the most important. Only in very specialized situations will it be more advantageous to use other radix-type FFTs.

- $N = r^v \rightarrow$ Radix – r FFTs. The special case r = 3 and r = 4 are not uncommon.

We'll focus on this type only in this course

- More generally, $N = p_1 p_2 p_3 \ldots p_l$ where the $p_i s$ are prime numbers lead to so called mixed-radix FFTs.

Radix-2 FFT

- We only consider radix – 2 FFTs (i.e., $N = 2^v$), where
  - DFT_N is decomposed into a cascade of v stages
  - Each stage is made up of $N/2$ DFT_2

Radix – 2 FFT via Decimation in Time:

- Let $x[n]$ be a discrete-time signal defined for $0 \leq n \leq N\text{-}1$, where $N = 2^v$.
- The basic idea behind decimation in time (DIT) is to partition the input sequence $x[n]$, of length N, into two sub-sequences, i.e. $x[2r]$ and $x[2r+1]$, $r = 0, 1, \ldots, (N/2) - 1$, corresponding to even and odd values of time, respectively.
- The N-point DFT of $x[n]$ can be computed by properly combining the $(N/2)$-point DFTs of each subsequences.
- In turn, the same principle can be applied in the computation of the $(N/2)$-point DFT of each subsequence, which can be reduced to DFTs of size $N/4$.
- This basic principle is repeated until only 2-point DFTs are involved.
- The final result is an FFT algorithm of complexity $N/2\log_2 N$ complex multiplication and $N\log_2 N$ complex additions..

Radix-2 FFT

- Radix-2 rearranges the DFT equation into 2 parts having indices as

$$X(k)=\sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

$$n=\{0,2,4,\dots N-2\}$$

$$n=\{1,3,5,\dots N-1\}$$

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x(2n)e^{-\frac{j2\pi k(2n)}{N}}+\sum_{n=0}^{\frac{N}{2}-1} x(2n+1)e^{-\frac{j2\pi k(2n+1)}{N}}$$

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x(2n)e^{-\frac{j2\pi kn}{\frac{N}{2}}}+e^{-\frac{j2\pi k}{N}}\sum_{n=0}^{\frac{N}{2}-1} x(2n+1)e^{-\frac{j2\pi kn}{\frac{N}{2}}}$$

$$X(k)=G(k)+W_N^k H(k)$$

- This is called Decimation in time because the time samples are rearranged in alternating groups

Radix-2 FFT

- Radix-2 rearranges the DFT equation into 2 parts having indices as

$$X(k)=\sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

$$n=\{0,2,4,\dots N-2\}$$
$$n=\{1,3,5,\dots N-1\}$$

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x(2n)e^{-\frac{j2\pi k(2n)}{N}}+\sum_{n=0}^{\frac{N}{2}-1} x(2n+1)e^{-\frac{j2\pi k(2n+1)}{N}}$$

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x(2n)e^{-\frac{j2\pi kn}{\frac{N}{2}}}+e^{-\frac{j2\pi k}{N}}\sum^{\frac{N}{2}-1} \cdots e^{-\frac{j2\pi kn}{N}}$$

$$X(k)=G(k)+W_N^kH(k)$$

- This is called Decimation in
samples are rearranged in a

The mathematical simplification reveal that all DFT freq: o/ps X(k) can be computed as the sum of the o/ps of two length N/2 DFTs, of even & odd indexed discrete time samples respectively, where the odd-indexed short DFT is multiplied by a so called Twiddle factor term.

2-Points FFT

The 2-point FFT:

- In the case $N = 2$, (1) specializes to,

$$X[k] = G[k] + H[k]W_2^k, \qquad k = 0,1$$

- Since, $W_2 = e^{-j\pi} = -1$, this can be further simplified to

$$X[0] = G[0] + H[1]$$

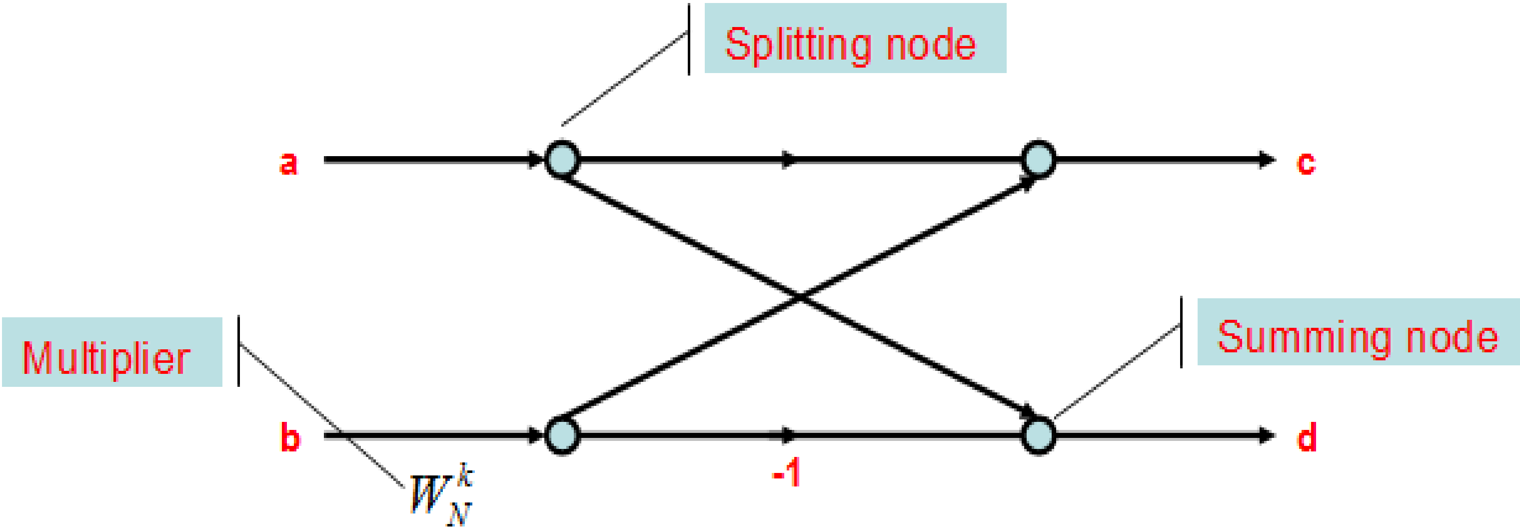
$$X[1] = G[0] - H[1]$$

Main steps of DIT:

- Split the summation $\sum_n$ in (1) into even $\sum_{n \text{ even}}$ and odd $\sum_{n \text{ odd}}$ parts as (N/2)-point DFTs.
- If N /2 = 2 stop; else, repeat the above steps for each of the individual (N/2)-point DFT.

"Butterfly" Signal Flow Graph

- In general, the equations for FFT are awkward to write mathematically, and therefore the algorithm is very often represented as a "butterfly" based signal flow graph (SFG), the butterfly being a simple SFG of the form:



- The multiplier is a complex number and the input data, a and b, may also be complex. One butterfly computation requires one complex multiply and two complex additions (assuming data is complex).

The 4-point FFT

Case $N = 4 = 2^2$:

- Step – 1: $$X[k] = X[0] + X[1]W_4^k + X[2]W_4^{2k} + X[3]W_4^{3k},$$ $$= \left(X[0] + X[2]W_4^{2k}\right) + W_4^k\left(X[1] + X[3]W_4^{2k}\right)$$

- Step – 2: Using the property $W_4^{2k} = W_4^k$, we can write $$X[k] = \left(X[0] + X[2]W_4^k\right) + W_4^k\left(X[1] + X[3]W_4^k\right)$$ $$= G[k] + W_4^k H[k]$$ $$G[k] = DFT_2\{\text{even samples}\}$$ $$H[k] = DFT_2\{\text{odd samples}\}$$

Note that $G[k]$ and $H[k]$, are 2-periodic, i.e.

$$G[k+2] = G[k], \qquad H[k+2] = H[k]$$

- Step – 3: Since $N/2 = 2$, we simply stop; that is, the 2-point DFTs $G[k]$ and $H[k]$ cannot be further simplified via DIT.

The 4-point FFT

Interpretation:

- The 4-point DFT can be computed by properly combining the 2-point DFTs of the even and odd samples, i.e. $G[k]$ and $H[k]$, respectively:

$$X[k] = G[k] + W_4^k H[k], \qquad k = 0, 1, 2, 3$$

- Since $G[k]$ and $H[k]$ are 2-periodic, they only need to be computed for $k = 0$, 1:

$$X_0[k] = G[0] + W_4^0 H[0]$$

$$X_1[k] = G[1] + W_4^1 H[1]$$

$$X_2[k] = G[2] + W_4^2 H[2] = G[0] + W_4^2 H[0]$$

$$X_3[k] = G[3] + W_4^3 H[0] = G[1] + W_4^3 H[1]$$

Radix-4 FFT

- The radix-4 decimation in time algorithm rearranges of every fourth discrete time index $n = \{0, 4, 8, \ldots N-4\}$

$$n = \{1, 5, 9, \ldots N-3\}$$

$$n = \{2, 6, 10, \ldots N-4\}$$

$$n = \{3, 7, 11, \ldots N-4\}$$

- This works out only when the FFT length is multiple of four.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}}$$

$$X(k) = \sum_{n=0}^{\frac{N}{4}-1} x(4n) e^{-\frac{j2\pi k(4n)}{N}} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+1) e^{-\frac{j2\pi k(4n+1)}{N}}$$

$$+ \sum_{n=0}^{\frac{N}{4}-1} x(4n+2) e^{-\frac{j2\pi k(4n+2)}{N}} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+3) e^{-\frac{j2\pi k(4n+3)}{N}}$$

Radix-4 FFT

$$X(k) = DFT_{\frac{N}{4}}[x(4n)] + W_N^k DFT_{\frac{N}{4}}[x(4n+1)]$$

$$+ W_N^{2k} DFT_{\frac{N}{4}}[x(4n+2)] + W_N^{3k} DFT_{\frac{N}{4}}[x(4n+3)]$$

- This is called Decimation in time because time samples are rearranged in alternating groups and a radix-4 algorithm because there are four groups.

Split Radix FFT

- By mixing radix-2 & radix-4 computations appropriately, an algorithm of lower complexity than other can be derived.

$$X(k)=\sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x(2n)e^{-\frac{j2\pi k(2n)}{N}}+\sum_{n=0}^{\frac{N}{4}-1} x(4n+1)e^{-\frac{j2\pi k(4n+1)}{N}}+\sum_{n=0}^{\frac{N}{4}-1} x(4n+3)e^{-\frac{j2\pi k(4n+3)}{N}}$$

$$X(k)=DFT_{\frac{N}{2}}[x(2n)]+W_N^k DFT_{\frac{N}{4}}[x(4n+1)]+W_N^{3k} DFT_{\frac{N}{4}}[x(4n+3)]$$

End of Chapter